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Hydromagnetic Stokes flow past a rotating sphere

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In the present investigation we consider hydromagnetic Stokes flow past a rotating sphere. The magnetic field is produced by a magnetic pole placed at the centre of the sphere. The problem is analysed by a combination of perturbation and numerical methods. It is seen that the flow reversal (due to rotation) at the rear portion of the sphere is enhanced as the strength of the magnetic field increases. In addition, we obtain the simultaneous effects of rotation and a magnetic field on the streamlines.

1. Introduction

The study of slow motion of a viscous incompressible fluid past a sphere has been a topic of interest for many years. Under the assumption that the Reynolds number $R \ll 1$, Stokes (1845) obtained an exact solution for the flow past a sphere due to a uniform stream at infinity. On the other hand Rubinow & Keller (1961) discussed the flow due to a spinning sphere which is moving through a quiescent incompressible viscous fluid, while Childress (1964) investigated the flow generated by motion of a sphere through a rotating fluid. Subsequently, Ranger (1971) examined slow flow past a rotating sphere with a uniform stream at infinity. On the basis of a linear superposition of the primary Stokes flow past a non-rotating sphere and a secondary flow induced by the spinning sphere, he showed that there is a region of reversed flow attached to the rear portion of the sphere. In a recent paper, Singh (1975) examined the flow past a sphere in the case when the sphere and the fluid at infinity are both rotating.

Though generalization of this class of problems to hydromagnetics attracted the attention of researchers some time ago, the literature indicates that only a few problems have been discussed so far. Most investigations in this area are concerned with inviscid hydromagnetic flow over a sphere. Nevertheless, Barthel & Lykoudis (1960) have examined the low Reynolds number flow past a sphere in the presence of a magnetic field produced by a dipole situated at the centre of the sphere. Confining their analysis to the case of small Hartmann number M $[= H_0 a(\mu_e/\rho\nu\eta)^{\frac{1}{2}}]$, they obtained a solution which represents small perturbations to classical Stokes flow. Later Riley (1961) devoted his attention to the hydromagnetic Stokes flow past a sphere when the magnetic field arises owing to a magnetic pole placed at the centre of the sphere. Assuming that not only the Reynolds number R but also the magnetic Reynolds number R_M was small, he obtained a series solution by the method of Frobenius. In view of the complexity of the analysis, he derived asymptotic solutions for large and small values of M and obtained a numerical solution for intermediate values of M. His solution reveals that the drag on the sphere increases with the Hartmann number and that the streamlines of the flow are significantly affected as the component of velocity normal to the magnetic lines of force is destroyed.

In the present paper we study the effect of a radial magnetic field on the flow past a slowly rotating sphere. Our aim is not only to obtain the simultaneous effects of rotation and a magnetic field, but also to find out what will happen to the flow reversal at the rear portion of the sphere, a result seen in the hydrodynamic case and discussed by Ranger (1971).

2. Formulation of the problem

Consider hydromagnetic flow of an electrically conducting, incompressible, viscous fluid past a non-conducting rotating sphere of radius a. Let U_0 be the velocity of the free stream at infinity and ω_1 be the angular velocity of the sphere about a diameter parallel to U_0. Taking the centre of the sphere as the origin, let us introduce a spherical polar co-ordinate system $(\bar{r}, \theta, \phi)$ in which θ is measured from the forward stagnation point. The velocity and the magnetic field components describing the flow are (q_r, q_θ, q_ϕ) and $(\bar{H}_r, \bar{H}_\theta, \bar{H}_\phi)$, where all these physical variables are functions of $\bar{r}$ and θ only. The equations governing the flow are obtained in non-dimensional form as

$$\begin{aligned}\frac{\partial}{\partial\theta}\left\{R(\mathbf{u}.\nabla)u_r - R\frac{u_\theta^2}{r} - \frac{T_1^2}{R}\frac{u_\phi^2}{r}\right\} - \frac{\partial}{\partial r}\left\{Rr(\mathbf{u}.\nabla)u_\theta - \frac{T_1^2}{R}u_\phi^2\cot\theta + Ru_r u_\theta\right\}\\ = \frac{\partial}{\partial\theta}\left(\nabla^2 u_r - \frac{2u_r}{r^2} - \frac{2\cot\theta}{r^2}u_\theta - \frac{2}{r^2}\frac{\partial u_\theta}{\partial\theta}\right) - \frac{\partial}{\partial r}\left(r\nabla^2 u_\theta - \frac{u_\theta}{r\sin^2\theta} + \frac{2}{r}\frac{\partial u_r}{\partial\theta}\right)\\ - \frac{M^2}{R_M}\left[\frac{\partial}{\partial\theta}\left\{\frac{T_1^2}{R^2}H_\phi\frac{\partial}{\partial r}(rH_\phi) + H_\theta\left(\frac{\partial}{\partial r}(rH_\theta) - \frac{\partial H_r}{\partial\theta}\right)\right\}\right.\\ \left. + \frac{\partial}{\partial r}\left\{H_r\left(\frac{\partial}{\partial r}(rH_\theta) - \frac{\partial H_r}{\partial\theta}\right) - \frac{T_1^2}{R^2\sin\theta}H_\phi\frac{\partial}{\partial\theta}(\sin\theta H_\phi)\right\}\right],\end{aligned} \tag{2.1}$$

$$\begin{aligned}R\left[(\mathbf{u}.\nabla)u_\phi + \frac{u_r u_\phi}{r} + \frac{u_\theta u_\phi}{r}\cot\theta\right]\\ = \nabla^2 u_\phi - \frac{u_\phi}{r^2\sin^2\theta} + \frac{1}{r\sin\theta}\frac{M^2}{R_M}\left(H_\theta\frac{\partial}{\partial\theta}(\sin\theta H_\phi) + H_r\frac{\partial}{\partial r}(r\sin\theta H_\phi)\right),\end{aligned} \tag{2.2}$$

$$\frac{R_M}{r\sin\theta}\frac{\partial}{\partial\theta}\{\sin\theta(u_r H_\theta - u_\theta H_r)\} = -\left(\nabla^2 H_r - \frac{2H_r}{r^2} - \frac{2\cot\theta}{r^2}H_\theta - \frac{2}{r^2}\frac{\partial H_\theta}{\partial\theta}\right), \tag{2.3}$$

$$\frac{R_M}{r}\left[\frac{\partial}{\partial r}\{r(u_\phi H_r - u_r H_\phi)\} - \frac{\partial}{\partial\theta}(u_\theta H_\phi - u_\phi H_\theta)\right] = -\left(\nabla^2 H_\phi - \frac{H_\phi}{r^2\sin^2\theta}\right), \tag{2.4}$$

$$\frac{\partial}{\partial r}(r^2\sin\theta u_r) + \frac{\partial}{\partial\theta}(r\sin\theta u_\theta) = 0, \tag{2.5}$$

$$\frac{\partial}{\partial r}(r^2\sin\theta H_r) + \frac{\partial}{\partial\theta}(r\sin\theta H_\theta) = 0, \tag{2.6}$$

where
$$u_r = \frac{q_r}{U_0},\quad u_\theta = \frac{q_\theta}{U_0},\quad u_\phi = \frac{q_\phi}{a\omega_1},\quad \mathbf{u} = (u_r, u_\theta, u_\phi),$$
$$r = \frac{\bar{r}}{a},\quad H_r = \frac{\bar{H}_r}{H_0},\quad H_\theta = \frac{\bar{H}_\theta}{H_0},\quad H_\phi = \frac{\bar{H}_\phi}{H_0}\frac{U_0}{a\omega_1},$$

in which H_0 is the strength of the magnetic field on the surface of the sphere. In the above equations $R\,(=aU_0/\nu)$ is the Reynolds number, $T_1\,(=a^2\omega_1/\nu)$ is the rotation parameter, $R_M\,(=aU_0/\eta)$ is the magnetic Reynolds number and $M^2\,(=\mu_e H_0^2 a^2/\rho\nu\eta)$ is the square of the Hartmann number. Equation (2.1) is obtained by eliminating the pressure between the first and second components of the momentum equation. The induction equation governing H_θ is not included as this equation depends upon (2.3) and (2.6). Once H_r is known, H_θ can be calculated from (2.6).

The boundary conditions at infinity are

$$u_r = -\cos\theta, \quad u_\theta = \sin\theta, \quad u_\phi = 0, \quad H_r = H_\theta = H_\phi = 0. \tag{2.7}$$

On the sphere, we have

$$u_r = u_\theta = 0, \quad u_\phi = 1 \tag{2.8}$$

and the magnetic field $\mathbf{H} = (H_r, H_\theta, H_\phi)$ is continuous with the field inside the sphere. The continuity of the magnetic field is valid if it is assumed that the magnetic permeability takes the same value everywhere.

The equations governing the magnetic field inside the sphere, in which no currents flow, are

$$\left.\begin{aligned} \frac{\partial}{\partial\theta}(\sin\theta H_\phi) = 0, \quad \frac{\partial}{\partial r}(rH_\phi) = 0, \\ \frac{\partial}{\partial r}(rH_\theta) - \frac{\partial H_r}{\partial\theta} = 0, \end{aligned}\right\} \tag{2.9}$$

and

$$\frac{\partial}{\partial r}(r^2\sin\theta H_r) + \frac{\partial}{\partial\theta}(r\sin\theta H_\theta) = 0. \tag{2.10}$$

Equations (2.1)–(2.6) and the boundary conditions (2.7)–(2.10) indicate that the problem depends upon four independent parameters, namely R, R_M, M^2 and T_1^2/R. In view of the complexity of the analysis, we consider the solution in the case in which $R, R_M \to 0$ while T_1^2/R and M^2 are $O(1)$. Since R_M is assumed to be small, we take the solutions to be of the form

$$\left.\begin{aligned} (u_r, u_\theta, u_\phi) &= (u_r^0, u_\theta^0, u_\phi^0) + R_M(u_r^1, u_\theta^1, u_\phi^1) + \dots, \\ (H_r, H_\theta, H_\phi) &= (H_r^0, H_\theta^0, H_\phi^0) + R_M(H_r^1, H_\theta^1, H_\phi^1) + \dots. \end{aligned}\right\} \tag{2.11}$$

Now let us find the distribution of the magnetic field inside the sphere. On substituting (2.11) into (2.9) and (2.10), we get the following zeroth- and first-order approximations:

$$\left.\begin{aligned} \frac{\partial}{\partial\theta}(\sin\theta H_\phi^0) = 0, \quad \frac{\partial}{\partial r}(rH_\phi^0) = 0, \\ \frac{\partial}{\partial r}(rH_\theta^0) - \frac{\partial H_r^0}{\partial\theta} = 0, \\ \frac{\partial}{\partial r}(r^2\sin\theta H_r^0) + \frac{\partial}{\partial\theta}(r\sin\theta H_\theta^0) = 0 \end{aligned}\right\} \tag{2.12}$$

and

$$\frac{\partial}{\partial\theta}(\sin\theta H_\phi^1) = 0, \quad \frac{\partial}{\partial r}(rH_\phi^1) = 0, \tag{2.13}$$

$$\frac{\partial}{\partial r}(rH_\theta^1) - \frac{\partial H_r^1}{\partial\theta} = 0, \tag{2.14}$$

$$\frac{\partial}{\partial r}(r^2\sin\theta H_r^1) + \frac{\partial}{\partial\theta}(r\sin\theta H_\theta^1) = 0. \tag{2.15}$$

As there is a magnetic pole at the centre of the sphere, equations (2.12) yield a solution of the form

$$(H_r^0, H_\theta^0, H_\phi^0) = (r^{-2}, 0, 0).$$

Now from (2.13) we get
$$H_\phi^1 = C/r\sin\theta, \tag{2.16}$$

where C is an arbitrary constant. On introducing a potential function Φ such that

$$H_r^1 = -\partial\Phi/\partial r, \quad H_\theta^1 = -r^{-1}\partial\Phi/\partial\theta,$$

(2.15) can be rewritten as

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right) = 0.$$

On solving for Φ, we have

$$\Phi = \sum_{n=0}^{\infty}\left(a_n r^n+\frac{b_n}{r^{n+1}}\right)P_n(\cos\theta). \tag{2.17}$$

In view of the singularity at $r = 0$, from (2.16) and (2.17) we find that the magnetic field components are

$$\left.\begin{aligned} H_r^1 &= \sum_{n=1}^{\infty} na_n r^{n-1}P_n(\cos\theta),\\ H_\theta^1 &= -\sum_{n=0}^{\infty} a_n r^n P_n'(\cos\theta)\sin\theta \end{aligned}\right\} \tag{2.18}$$

and
$$H_\phi^1 = 0.$$

Now, on introducing (2.11) into equations (2.1)–(2.6) and boundary conditions (2.7) and (2.8), we get systems of equations and boundary conditions corresponding to different approximations.

In the zeroth approximation, the equations governing the magnetic induction are

$$\nabla^2 H_r^0-\frac{2H_r^0}{r^2}-\frac{2\cot\theta}{r^2}H_\theta^0-\frac{2}{r^2}\frac{\partial H_\theta^0}{\partial\theta} = 0,$$

$$\nabla^2 H_\phi^0 - H_\phi^0/r^2\sin^2\theta = 0,$$

$$\frac{\partial}{\partial r}(r^2\sin\theta H_r^0)+\frac{\partial}{\partial\theta}(r\sin\theta H_\theta^0) = 0.$$

The corresponding boundary conditions are

$$H_r^0 = 1, \quad H_\theta^0 = H_\phi^0 = 0 \quad \text{at} \quad r = 1$$

and
$$H_r^0 = H_\theta^0 = H_\phi^0 = 0 \quad \text{at} \quad r = \infty.$$

On solving the above boundary-value problem, we get

$$(H_r^0, H_\theta^0, H_\phi^0) = (r^{-2}, 0, 0).$$

This solution indicates that the magnetic field inside the flow is the same as that within the sphere. This is due to the fact that in the zeroth approximation there is no interaction between the flow and the magnetic field. We also note that the zeroth approximations of the momentum equations are identically satisfied.

For the first approximation, the equations of motion can be written as

$$\frac{\partial}{\partial\theta}\left\{R(\mathbf{u}^0.\nabla)\,u_r^0-R\frac{u_\theta^{02}}{r}-\frac{T_1^2}{R}\frac{u_\phi^{02}}{r}\right\}-\frac{\partial}{\partial r}\left\{Rr(\mathbf{u}^0.\nabla)\,u_\theta^0-\frac{T_1^2}{R}u_\phi^{02}\cot\theta+Ru_r^0u_\theta^0\right\}$$

$$=\frac{\partial}{\partial\theta}\left(\nabla^2u_r^0-\frac{2u_r^0}{r^2}-\frac{2\cot\theta}{r^2}u_\theta^0-\frac{2}{r^2}\frac{\partial u_\theta^0}{\partial\theta}\right)-\frac{\partial}{\partial r}\left(r\nabla^2u_\theta^0-\frac{u_\theta^0}{r\sin^2\theta}+\frac{2}{r}\frac{\partial u_r^0}{\partial\theta}\right)$$

$$-M^2\frac{\partial}{\partial r}\left\{\frac{1}{r^2}\left(\frac{\partial}{\partial r}(rH_\theta^1)-\frac{\partial H_r^1}{\partial\theta}\right)\right\}, \tag{2.19}$$

$$R\left\{(\mathbf{u}^0.\nabla)\,u_\phi^0+\frac{u_r^0u_\phi^0}{r}+\frac{u_\theta^0u_\phi^0}{r}\cot\theta\right\}=\nabla^2u_\phi^0-\frac{u_\phi^0}{r^2\sin^2\theta}+\frac{M^2}{r^3}\frac{\partial}{\partial r}(rH_\phi^1), \tag{2.20}$$

$$\frac{1}{r^3\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta u_\theta^0)=\nabla^2H_r^1-\frac{2H_r^1}{r^2}-\frac{2\cot\theta}{r^2}H_\theta^1-\frac{2}{r^2}\frac{\partial H_\theta^1}{\partial\theta}, \tag{2.21}$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{u_\phi^0}{r}\right)=-\left(\nabla^2H_\phi^1-\frac{H_\phi^1}{r^2\sin^2\theta}\right), \tag{2.22}$$

$$\frac{\partial}{\partial r}(r^2\sin\theta u_r^0)+\frac{\partial}{\partial\theta}(r\sin\theta u_\theta^0)=0 \tag{2.23}$$

and

$$\frac{\partial}{\partial r}(r^2\sin\theta H_r^1)+\frac{\partial}{\partial\theta}(r\sin\theta H_\theta^1)=0. \tag{2.24}$$

The corresponding boundary conditions at $r=\infty$ are

$$u_r^0=-\cos\theta,\quad u_\theta^0=\sin\theta,\quad u_\phi^0=0,\quad H_r^1=H_\theta^1=H_\phi^1=0$$

while at $r=1$

$$u_r^0=u_\theta^0=0,\quad u_\phi^0=1,\quad H_\phi^1=0$$

and H_r^1 and H_θ^1 are continuous with the first approximation of the magnetic field inside the sphere.

As the motion under consideration is two-dimensional, we introduce stream functions ψ_0 and ψ_1 for the velocity and magnetic field respectively. Thus we have

$$u_r^0=\frac{1}{r^2\sin\theta}\frac{\partial\psi_0}{\partial\theta},\quad u_\theta^0=-\frac{1}{r\sin\theta}\frac{\partial\psi_0}{\partial r},$$

$$H_r^1=\frac{1}{r^2\sin\theta}\frac{\partial\psi_1}{\partial\theta},\quad H_\theta^1=-\frac{1}{r\sin\theta}\frac{\partial\psi_1}{\partial r}.$$

Now, following the analysis of Ranger (1971), the solutions for the present problem are taken as

$$\psi_0=r^2\sin^2\theta[\tfrac{1}{2}f+(T_1^2/R)\,F\cos\theta], \tag{2.25}$$

$$\psi_1=r^2\sin^2\theta[\tfrac{1}{2}g+(T_1^2/R)\,G\cos\theta], \tag{2.26}$$

$$\Omega=x+\tfrac{1}{2}RX\cos\,\theta \tag{2.27}$$

and

$$\chi=y+\tfrac{1}{2}RY\cos\theta, \tag{2.28}$$

where $\Omega = ru_\phi^0/\sin\theta$, $\chi = rH_\phi^1/\sin\theta$, and f, F, g, G, x, X, y and Y are functions of r only. The expressions for ψ_0 and ψ_1 are the zeroth approximations of the perturbation solution with R as parameter, while the expressions for Ω and χ consist of both zeroth- and first-order terms in R.

With the above choice for ψ_0 and ψ_1, the velocity and magnetic field components u_r, u_θ, H_r and H_θ are given by

$$u_r = f\cos\theta + (T_1^2/R)\,F(3\cos^2\theta - 1), \tag{2.29}$$

$$u_\theta = \sin\theta[v + (T_1^2/R)\,V\cos\theta], \tag{2.30}$$

$$H_r = g\cos\theta + (T_1^2/R)\,G(3\cos^2\theta - 1) \tag{2.31}$$

and

$$H_\theta = -\sin\theta\{g + \tfrac{1}{2}rg' + (T_1^2/R)\cos\theta(2G + rG')\}, \tag{2.32}$$

where $v = -(f + \frac{1}{2}rf')$ and $V = -(2F + rF')$. On introducing (2.27)–(2.31) into (2.19)–(2.22) and then equating the corresponding terms on both sides, we get the following sets of simultaneous equations.

First system:

$$f^{\text{iv}} + \frac{8}{r}f''' + \frac{8}{r^2}f'' - \frac{8}{r^3}f' + \frac{M^2}{r^2}\left(g''' + \frac{4}{r}g'' - \frac{4}{r^2}g'\right) = 0, \tag{2.33}$$

$$g'' + \frac{4}{r}g' + \frac{f'}{r^2} + \frac{2f}{r^3} = 0, \tag{2.34}$$

$$\left(\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr}\right)\left(\frac{x}{r}\right) - \frac{2x}{r^3} + \frac{M^2}{r^3}\frac{dy}{dr} = 0, \tag{2.35}$$

$$\left(\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr}\right)\left(\frac{y}{r}\right) - \frac{2y}{r^3} + \frac{1}{r}\frac{d}{dr}\left(\frac{x}{r^2}\right) = 0. \tag{2.36}$$

Second system:

$$F^{\text{iv}} + \frac{8}{r}F''' - \frac{24}{r^3}F' + \frac{24}{r^4}F + \frac{M^2}{r^2}\left(G''' + \frac{4}{r}G'' - \frac{8}{r^2}G' + \frac{8}{r^3}G\right) = \frac{1}{r^2}\left(\frac{d}{dr} - \frac{2}{r}\right)\left(\frac{x^2}{r^2}\right), \tag{2.37}$$

$$G'' + \frac{4}{r}G' - \frac{4G}{r^2} + \frac{(2F + rF')}{r^3} = 0, \tag{2.38}$$

$$\left(\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr}\right)\left(\frac{X}{r}\right) - \frac{6X}{r^3} + \frac{M^2}{r^3}\frac{dY}{dr} = \frac{2fx'}{r} - \frac{2f'x}{r} - \frac{4fx}{r^2}, \tag{2.39}$$

$$\left(\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr}\right)\left(\frac{Y}{r}\right) + \frac{1}{r}\frac{d}{dr}\left(\frac{X}{r^2}\right) - \frac{6Y}{r^3} = 0. \tag{2.40}$$

Before we specify the boundary conditions corresponding to the above two systems, let us determine the boundary conditions on the magnetic field at $r = 1$. Demanding that the magnetic field components inside the sphere and in the flow are continuous at $r = 1$, we get

$$g(1) = a_1, \quad a_n = 0 \quad \text{for} \quad n > 1,$$

$$g'(1) = 0, \quad y(1) = 0, \quad G = G' = Y = 0.$$

Thus the boundary conditions appropriate for the problem are

$$\left.\begin{array}{l} f = f' = g' = y = 0, \quad x = 1 \quad \text{at} \quad r = 1 \\ f \to -1, \quad f', g' \to 0, \quad x/r, y/r \to 0 \quad \text{as} \quad r \to \infty \end{array}\right\} \text{for the first system} \tag{2.41}$$

and

$$\left.\begin{array}{l} F = F' = G = Y = X = 0 \quad \text{at} \quad r = 1 \\ F, F', G \to 0, \quad Y/r, X/r \to 0 \quad \text{as} \quad r \to \infty \end{array}\right\} \text{for the second system.} \tag{2.42}$$

Now, in view of (2.34) and (2.38), (2.33) and (2.37) can be rewritten as

$$f^{\text{iv}} + \frac{8}{r} f''' + \frac{8}{r^2} f'' - \frac{8}{r^3} f' - \frac{M^2}{r^2}\left(\frac{f''}{r^2} - \frac{6f}{r^4}\right) = 0 \tag{2.43}$$

and

$$F^{\text{iv}} + \frac{8}{r} F''' - \frac{24}{r^3} F' + \frac{24}{r^4} F - \frac{M^2}{r^4}\left(F'' - \frac{6F}{r^2}\right) = \frac{1}{r^2}\left(\frac{d}{dr} - \frac{2}{r}\right)\left(\frac{x^2}{r^2}\right). \tag{2.44}$$

Equations (2.35), (2.36) and (2.43) together with the boundary conditions

$$\left.\begin{array}{l} f = f' = 0, \quad x = 1, \quad y = 0 \quad \text{at} \quad r = 1, \\ f \to -1, \quad f' \to 0, \quad x/r, y/r \to 0 \quad \text{as} \quad r \to \infty \end{array}\right\} \tag{2.45}$$

constitute the first system.

The second system comprises (2.39), (2.40) and (2.44) with the boundary conditions

$$\left.\begin{array}{l} F = F' = X = Y = 0 \quad \text{at} \quad r = 1, \\ F, F' \to 0, \quad X/r, Y/r \to 0 \quad \text{as} \quad r \to \infty. \end{array}\right\} \tag{2.46}$$

On solving the above boundary-value problems, we get the velocity and the ϕ component of the induced magnetic field. Then the components of the induced magnetic field in the r and θ directions can be determined from (2.34) and (2.38). In the absence of rotation ($T_1 = 0$), the problem given by the first system reduces to that investigated by Riley. On the other hand, when the magnetic field is absent, both systems reduce to those of Ranger.

In what follows, we obtain the numerical solution of the boundary-value problems by using the Runge–Kutta–Merson method (see Lance 1960, p. 56), which has an additional formula for the determination of the truncation error. This facilitates a suitable choice of step length.

3. Solution of the problem

Let us introduce the transformation $Z = (r-1)/\delta$, where $\delta = 3M$. Now (2.43) can be written as the set of first-order equations

$$Dy_1 = y_2, \quad Dy_2 = y_3, \quad Dy_3 = y_4,$$

$$Dy_4 = -\frac{8}{Z+\delta^{-1}} y_4 - \frac{8}{(Z+\delta^{-1})^2} y_3 + \frac{8}{(Z+\delta^{-1})^3} y_2 + \frac{1}{9(Z+\delta^{-1})^2}\left\{\frac{y_3}{(Z+\delta^{-1})^2} - \frac{6y_1}{(Z+\delta^{-1})^4}\right\},$$

where $y_1 = f$. Combination of these equations with the two sets of initial conditions

$$\left.\begin{array}{l} y_1 = y_2 = 0, \quad y_3 = 1, \quad y_4 = 0 \\ y_1 = y_2 = y_3 = 0, \quad y_4 = 1 \end{array}\right\} \quad \text{at} \quad Z = 0$$

leads to a solution of the form $\quad f = A_1 f_1 + A_2 f_2,$

where f_1 and f_2 are the two particular solutions corresponding to the two sets of initial conditions. Use of the boundary conditions $f = -1$ and $f' = 0$ gives A_1 and A_2.

Now we convert (2.35) and (2.36) into the first-order equations

$$Dy_1 = y_2,$$

$$Dy_2 = -\frac{2}{Z+\delta^{-1}}y_2 + \frac{2}{(Z+\delta^{-1})^2}y_1 - \frac{M}{3(Z+\delta^{-1})^2}\left(y_4 + \frac{y_3}{Z+\delta^{-1}}\right) + \frac{2}{(Z+\delta^{-1})^2},$$

$$Dy_3 = y_4,$$

$$Dy_4 = -\frac{2}{Z+\delta^{-1}}y_4 + \frac{2}{(Z+\delta^{-1})^2}y_3 - \frac{1}{3M}\left\{\frac{y_2}{(Z+\delta^{-1})^2} - \frac{y_1}{(Z+\delta^{-1})^3}\right\} + \frac{1}{3M(Z+\delta^{-1})^3},$$

where $y_1 = x/r - 1$ and $y_3 = y/r$. These equations together with the three sets of initial conditions

$$\left.\begin{array}{l} y_1 = 0, \quad y_2 = 1, \quad y_3 = y_4 = 0 \\ y_1 = y_2 = y_3 = 0, \quad y_4 = 1 \\ y_1 = y_2 = y_3 = y_4 = 0 \end{array}\right\} \quad \text{at} \quad Z = 0$$

yield the solution

$$x/r - 1 = A_1 y_1^{(1)} + A_2 y_1^{(2)} + y_1^{(3)},$$

$$y/r = A_1 y_3^{(1)} + A_2 y_3^{(2)} + y_3^{(3)},$$

in which the superscripts (1)–(3) correspond to the three particular solutions. Applying the boundary conditions x/r, $y/r \to 0$ as $Z \to \infty$ determines the constants A_1 and A_2.

Following the above procedure, we next solve the second system for F, X and Y, thus obtaining the velocity and magnetic field components.

4. Discussion of the results

To gain deeper insight into the flow, we have computed the numerical solution for various values of M^2 and T_1^2/R. The functions f, F, v and V related to the velocity components u_r and u_θ [see (2.29) and (2.30)] are presented in figures 1 and 3. Streamline patterns exhibiting the effects of the magnetic field (M^2) and the rotation (T_1^2/R) are given in figures 2 and 4–6.

In the absence of rotation, (2.29) and (2.30) reduce to the form

$$u_r = f\cos\theta, \quad u_\theta = v\sin\theta.$$

As $f < 0$ and $v > 0$, $u_\theta > 0$ for $0 \leqslant \theta < \pi$ while $u_r \lesseqgtr 0$ for $0 \leqslant \theta < \frac{1}{2}\pi$ and $\frac{1}{2}\pi < \theta \leqslant \pi$ respectively.

From figure 1 we see that f and v decrease as the magnetic field increases. Hence we find that the flow along and across the magnetic field lines is restricted as the magnetic field is enhanced. The flow across the magnetic field lines is reduced because the Lorentz force, which acts in the $-\theta$ direction, opposes the motion. The inward and outward radial flow which takes place in the first and second quadrants, respectively, is curtailed as the pressure gradient, which changes sign from positive to negative, is increased by the magnetic field permeating the medium. This reduction in the radial flow can also be attributed to the fact that the magnetic field lines (originally in the radial direction) are displaced in the downstream direction.

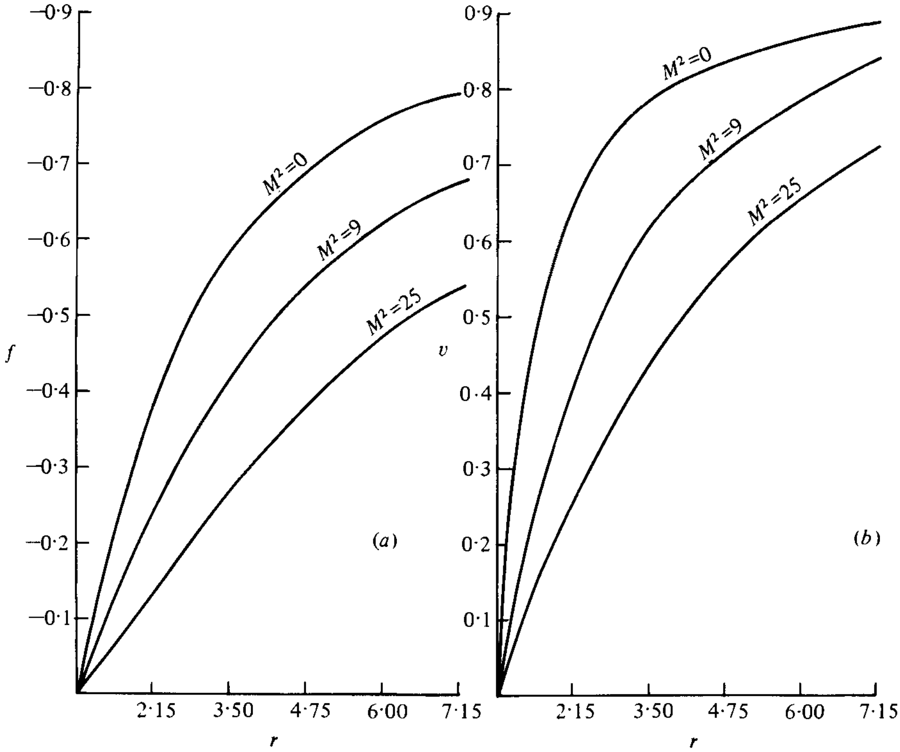

FIGURE 1. Effect of magnetic field on (*a*) f and (*b*) v $(T_1^2/R = 0)$.

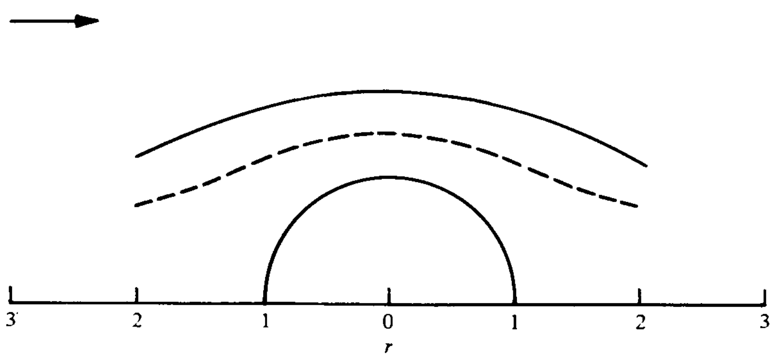

FIGURE 2. Streamlines showing the effect of a magnetic field $(T_1^2/R = 0,\ \psi = -0{\cdot}125)$. ——, $M^2 = 25$; - - -, $M^2 = 0$.

From figure 2 we see that the streamlines in the magnetic case ($M^2 = 25$) are pushed away from the sphere compared with those in Stokes flow ($M^2 = 0$). This is due to the fact that f (which occurs in the relation $\psi = \frac{1}{2} fr^2 \sin^2\theta$) is reduced by the presence of the magnetic field.

Now let us consider the case when the sphere is rotating. Here u_r and u_θ are given by

$$u_r = f\cos\theta + (T_1^2/R)\, F(3\cos^2\theta - 1), \tag{4.1}$$

$$u_\theta = \sin\theta[v + (T_1^2/R)\, V\cos\theta] \tag{4.2}$$

and

$$\psi = r^2\sin^2\theta[\tfrac{1}{2}f + (T_1^2/R)\, F\cos\theta]. \tag{4.3}$$

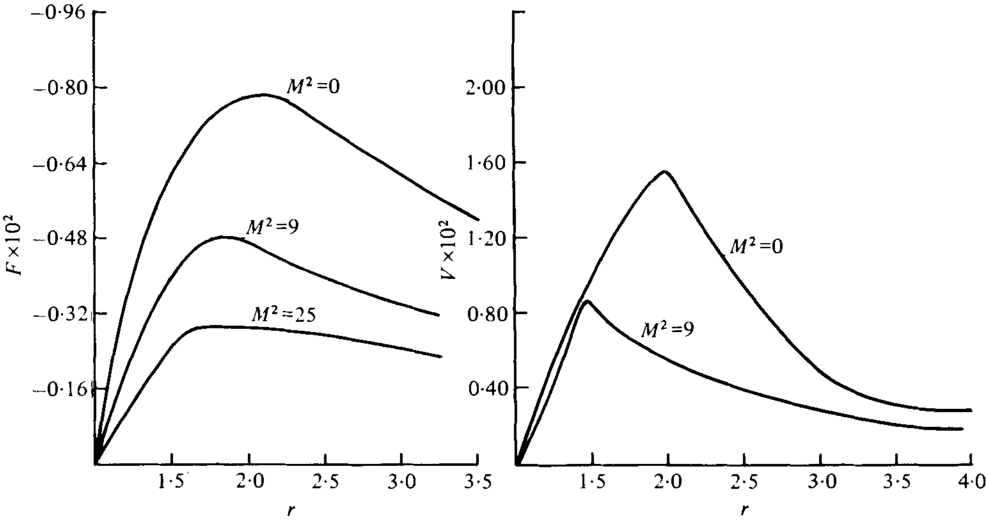

FIGURE 3. Effect of magnetic field on (*a*) F and (*b*) V $(T_1^2/R \neq 0)$.

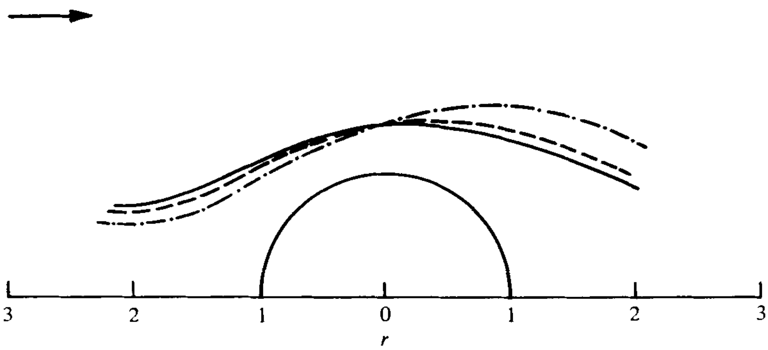

FIGURE 4. Streamlines showing the effect of rotation ($M = 0$, $\psi = -0{\cdot}125$). ——, $T_1^2/R = 5$; – – –, $T_1^2/R = 10$; —·—, $T_1^2/R = 15$.

In the absence of the magnetic field (see figures 1 and 3), it is clear that F and V are very much smaller than f and v. In view of (4.1), we see that as T_1^2/R increases the inward radial flow (in the first quadrant) increases or decreases according as

$$\cos\theta \lessgtr 1/\sqrt{3},$$

i.e. $\theta \lessgtr 54°\,42'$. On the other hand, the outward radial flow (in the second quadrant) increases or decreases according as $\theta \lessgtr 144°\,42'$. Further, from (4.2) we find that as the rotation increases the azimuthal flow increases for $0 \leqslant \theta < \frac{1}{2}\pi$ while it decreases for $\frac{1}{2}\pi < \theta \leqslant \pi$. The above behaviour of the velocity components can be understood by considering the effect of the centrifugal force produced by the rotation. As the sphere rotates with higher angular velocity, the fluid in the zone $54°\,42' \leqslant \theta \leqslant 144°\,42'$ is driven out by the centrifugal force. In order to compensate for this motion, the fluid in the other zones is sucked towards the poles of the sphere.

From figure 4 it is interesting to note that, in the front portion of the sphere, the streamlines are pulled towards the axis of rotation, while they are pushed away in the rear portion.

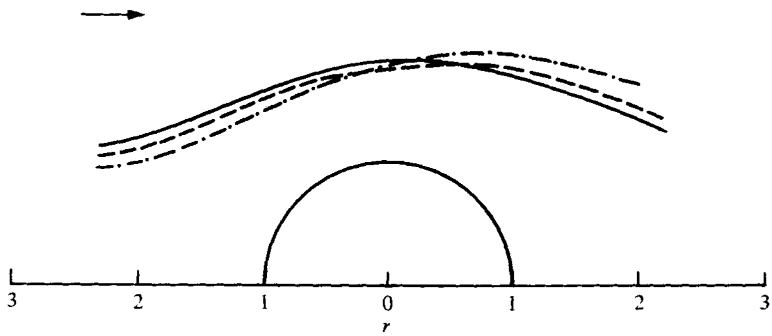

FIGURE 5. Streamlines showing the effect of rotation in the presence of a magnetic field ($M^2 = 25$, $\psi = -0{\cdot}125$). ——, $T_1^2/R = 5$; ---, $T_1^2/R = 10$; —·—, $T_1^2/R = 15$.

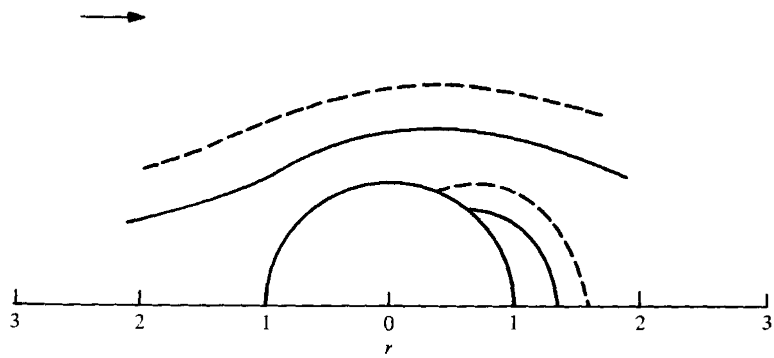

FIGURE 6. Streamlines and back-flow region ($T_1^2/R = 10$, $\psi = -0{\cdot}125$). ——, $M^2 = 0$; ---, $M^2 = 25$.

In the light of the above findings, let us examine the simultaneous effects of rotation and a magnetic field. From figure 3 we see that the profiles of F and V are very much flattened as the strength of the magnetic field increases. Thus, in view of (4.1) and (4.2), the effects of rotation are significantly counteracted by the presence of the magnetic field. As f and v are the dominant quantities in u_r and u_θ, it is to be expected that the magnetic field plays a vital role in determining the characteristics of the flow. Nevertheless, from figure 5 it is apparent that the rotation maintains its role of pulling down the upstream flow and pushing up the downstream flow. However, this effect is less prevalent as the magnetic field decelerates the flow considerably.

The expression (4.3) for ψ suggests that back flow arises for $\cos\theta = -fR/2T_1^2F$, if $T_1^2/R > f/2F$. From figure 6, it is worth noting that the reversed flow produced by rotation is enhanced by the presence of the magnetic field. The mechanism underlying this phenomenon can be explained as follows. The rotation produces a flow from the poles to the equatorial plane of the sphere. This flow is restricted by the axial flow, but when it dominates the axial flow, back flow occurs at the rear portion of the sphere. In the presence of a magnetic field this back flow is enhanced, and is expected to set in at a lower value of T_1^2/R, since the magnetic field opposes the axial flow.

Numerical solutions obtained for the particular cases (i) $T_1^2/R = 0$, $M \neq 0$ and (ii) $T_1^2/R \neq 0$, $M = 0$ are in good agreement with those discussed by Riley and Ranger.

From the above analysis, we conclude that the flow past a sphere is significantly affected by rotation of the sphere and a magnetic field permeating the medium.

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